

Virtually Semisimple Modules and a Generalization of the Wedderburn-Artin Theorem ^{*†‡}

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Abstract

By any measure, semisimple modules form one of the most important classes of modules and play a distinguished role in the module theory and its applications. One of the most fundamental results in this area is the Wedderburn-Artin theorem. In this paper, we establish natural generalizations of semisimple modules and give a generalization of the Wedderburn-Artin theorem. We study modules in which every submodule is isomorphic to a direct summand and name them *virtually semisimple modules*. A module ${}_R M$ is called *completely virtually semisimple* if each submodules of M is a virtually semisimple module. A ring R is then called *left (completely) virtually semisimple* if ${}_R R$ is a left (completely) virtually semisimple R -module. Among other things, we give several characterizations of left (completely) virtually semisimple rings. For instance, it is shown that a ring R is left completely virtually semisimple if and only if $R \cong \prod_{i=1}^k M_{n_i}(D_i)$ where $k, n_1, \dots, n_k \in \mathbb{N}$ and each D_i is a principal left ideal domain. Moreover, the integers k, n_1, \dots, n_k and the principal left ideal domains D_1, \dots, D_k are uniquely determined (up to isomorphism) by R .

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1 Introduction

In the study of k -algebras A (associative or non-associative) over a commutative ring k , the semisimplicity plays an important role. It is known that for a k -module M , any A -module structure corresponds to a k -algebra homomorphism $\theta : A \rightarrow \text{End}_k(M)$ with $\theta(a) =$ the multiplication map by $a \in A$ and $\ker \theta = \text{Ann}_A(M)$. Furthermore, if M is an A -module with $S = \text{End}_A(M)$, then the image of θ lies in $\text{End}_S(M)$. This shows that if either k is a division ring and A is a simple algebra or A has a simple faithful module, then A can be represented by linear transformations and in the finite dimension case, A is a subalgebra of an n -by- n matrix algebra over a division ring. The concept of semisimple module is often introduced as a direct sum of simple ones. E. Cartan characterized semisimple Lie algebras and shown that every finite-dimensional module over a semisimple Lie algebra with zero characteristic is a semisimple module (see for example [15, Page 27]). Finite dimensional algebras are serious examples of Artinian algebras (algebras with descending chain conditions on their left (right) ideals). They are considered as Artinian rings in the ring theory. Artinian rings with a faithful semisimple module are known to be *semisimple* rings that form a fundamental and important class of rings. If R is any ring, an R -module M is said to be *simple* in case $M \neq (0)$ and it has no non-trivial submodules. *Semisimple* R -modules are then considered as direct sums of simple R -modules. It is well known that R is a semisimple ring if and only if the left (right) R -module R is semisimple if and only if all left (right) R -modules are semisimple. As the historical perspective, the fundamental characterization of finite-dimensional k -algebras was originally done by Wedderburn in his 1907 paper ([20]). After that in 1927, E. Artin generalizes the Wedderburn's theorem for semisimple algebras ([2]). In fact, the Wedderburn-Artin's result is a landmark in the theory of noncommutative rings. We recall this theorem as follows:

Wedderburn-Artin Theorem: *A ring R is semisimple if and only if $R \cong \prod_{i=1}^k M_{n_i}(D_i)$ where $k, n_1, \dots, n_k \in \mathbb{N}$ and each D_i is a division ring. Moreover, the integers k, n_1, \dots, n_k and the division rings D_1, \dots, D_k are uniquely determined (up to a permutation).*

By the Wedderburn-Artin Theorem, the study of semisimple rings can be reduced to the study of modules over division rings. We note that a semisimple module is a type of module that can be understood easily from its parts. More precisely, a module M is semisimple if and only if every submodule of M is a direct summand. In this paper, we study modules (resp., rings) in which every submodule (resp., left ideal) is isomorphic to a direct summand. We will show that the study of such rings can be reduced to the study of modules over principal left ideal domains. This gives a generalization of the Wedderburn-Artin Theorem.

Throughout this paper, all rings are associative with identity and all modules are

unitary. Following [9], we denote by $\text{K.dim}(M)$ the *Krull dimension* of a module M . If $\alpha \geq 0$ is an ordinal number then the module M is said to be α -critical provided $\text{K.dim}(M) = \alpha$ while $\text{K.dim}(M/N) < \alpha$ for all non-zero submodules N of M . A module is called *critical* if it is α -critical for some ordinal $\alpha \geq 0$.

Definitions 1.1. We say that an R -module M is *virtually semisimple* if each submodule of M is isomorphic to a direct summand of M . If each submodule of M is a virtually semisimple module, we call M *completely virtually semisimple*. If ${}_R R$ is (resp., R_R) is a virtually semisimple module, we then say that R is a *left* (resp., *right*) *virtually semisimple ring*. A *left* (resp., *right*) *completely virtually simple ring* is similarly defined.

In Section 2, we introduce the fundamental tools of this study and give some basic properties of virtually semisimple modules. Among of other things, we show for a non-zero virtually semisimple module ${}_R M$ the following statements are equivalent: (1) ${}_R M$ is finitely generated; (2) ${}_R M$ is Noetherian; (3) $\text{u.dim}({}_R M) < +\infty$, and (4) $M \cong R/P_1 \oplus \dots \oplus R/P_n$ where $n \in \mathbb{N}$, each P_i is a quasi prime left ideal of R such that R/P_i is a critical Noetherian R -module (here, $\text{u.dim}({}_R M)$ is the uniform dimension of the module ${}_R M$ and we say that a left ideal P of a ring R is *quasi prime* if $P \neq R$ and, for ideals $A, B \subseteq R$, $AB \subseteq P \subseteq A \cap B$ implies that $A \subseteq P$ or $B \subseteq P$) (see Proposition 2.7). Also, it is shown that a finitely generated quasi projective 1-epi-retractable R -module M is virtually semisimple if and only if $\text{End}_R(M)$ is a semiprime principal left ideal ring (Theorem 2.9). An R -module M is called (resp., *n-epi-retractable*) *epi-retractable* if for every (resp., n -generated) submodule N of M there exists an epimorphism $f : M \rightarrow N$. This concept is studied in [6].

Section 3 is devoted to study of the structure of left (completely) virtually semisimple rings. We give several characterizations of left virtually semisimple rings in Theorem 3.4. We shall give some examples to show that the (completely) virtually semisimple are not symmetric properties for a ring, and also completely virtually semisimple modules properly lies between the class of semisimple modules and the class of virtually semisimple modules; see Examples 3.7~3.10. While the left virtually semisimple is not a Morita invariant ring property, we proved that the left completely virtually semisimple is (see Proposition 3.3). In Theorem 3.13, we will give the following generalization of the Wedderburn-Artin theorem:

A Generalization of the Wedderburn-Artin Theorem: *A ring R is left completely virtually semisimple if and only if $R \cong \prod_{i=1}^k M_{n_i}(D_i)$ where $k, n_1, \dots, n_k \in \mathbb{N}$ and each D_i is a principal left ideal domain. Moreover, the integers k, n_1, \dots, n_k and the principal left ideal domains D_1, \dots, D_k are uniquely determined (up to isomorphism) by R .*

Any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [1] and [13].

2 Virtually semisimple modules

The subject of our study in this section is some basic properties of virtually semisimple modules. We introduce the fundamental tools of this study for latter uses.

A module ${}_R M$ is said to be *Dedekind finite* if $M = M \oplus N$ for some R -module N , then $N = 0$. Let M and P be R -modules. We recall that P is *M-projective* if every diagram in $R\text{-Mod}$ with exact row

$$\begin{array}{ccc} & P & \\ & \downarrow & \\ M & \longrightarrow & N \longrightarrow 0 \end{array}$$

can be extended commutatively by a morphism $P \longrightarrow M$. Also, if P is P -projective, then P is called *quasi projective*. A direct summand K of M is denoted by $K \leq^\oplus {}_R M$.

Proposition 2.1. *The following statements hold.*

- (i) *Let ${}_R M$ be a non-zero virtually semisimple module. If $M = M_1 \oplus M_2$ is a decomposition for ${}_R M$ such that $\text{Hom}_R(M_1, M_2) = 0$ then M_2 is a virtually semisimple R -module.*
- (ii) *If I is an ideal of R with $IM = 0$. Then M is a virtually semisimple (R/I) -module if and only if ${}_R M$ is virtually semisimple.*
- (iii) *A module ${}_R M$ is virtually semisimple quasi projective if and only if it is an epi-retractable R -module and all of its submodules are M -projective.*
- (iv) *Being (completely) virtually semisimple module is a Morita invariant property.*
- (v) *Let ${}_R M$ be virtually semisimple and $W \leq {}_R M$. If W contains any submodule K of M with K is embedded in W , then ${}_R W$ is virtually semisimple and there is a direct summand K of ${}_R M$ such that $K \cong W$ and $K \oplus K' = W$ for some submodule K' . In particular, if ${}_R W$ is Dedekind finite, then $W \leq^\oplus M$.*

Proof. (i) Let $K \leq M_2$. Since ${}_R M$ is a virtually semisimple module, there is a decomposition $N \oplus N' = M$ with $N \cong K$. It easily seen that M_1 is fully invariant submodule of M because $\text{Hom}_R(M_1, M_2) = 0$. It follows that $M_1 = (N \cap M_1) \oplus (N' \cap M_1)$ and hence $N \cap M_1 = 0$ because $\text{Hom}_R(M_1, N) = 0$. Thus $M_1 \subseteq N'$ which implies that $N' = M_1 \oplus (N' \cap M_2)$. Now we have $M = N \oplus M_1 \oplus (N' \cap M_2) = M_1 \oplus M_2$. It follows that N and hence K is isomorphic to a direct summand of M_2 , as desired.

(ii) This is routine.

(iii) The necessity is clear by the definition and [21, Proposition 18.2]. Conversely, assume that ${}_R M$ is epi-retractable and every submodule of M is M -projective. If $N \leq M$, then there is a surjective homomorphism $f : M \longrightarrow N$. Since now N is M -projective, $\text{Ker} f$ must be a direct summand of M , the proof is complete.

(iv) It is shown that Morita equivalences preserve monomorphisms and direct sums (see for instance [1, §21]).

(v) If $W \leq_R M$ as stated in the above, then by our assumption, $W \cong K$ where $K \leq^\oplus_R M$ and $K \subseteq W$. It follows that K is also a direct summand of W . Similarly, if $N \leq W$ and $V \oplus V' = M$ with $V \cong N$, we can deduce that N is isomorphic to a direct summand of W , that is ${}_R W$ is virtually semisimple. The last statement is now clear. \square

Let R_1 and R_2 be rings and $T = R_1 \oplus R_2$. It is well-known that any T -module M has the form $M_1 \oplus M_2$, for some R_i -modules $M_i (i=1, 2)$. In fact, $M = e_1 M \oplus e_2 M$ where e_1 and e_2 are central orthogonal idempotents in T such that $e_1 R_2 = e_2 R_2 = 0$ and $e_1 + e_2 = 1_T$. Clearly $e_i M$ is naturally an R_i -module (as well as T -module) for $i=1, 2$. This shows that $\text{Hom}_T(M_i, M_j) = 0$ for $i \neq j$. Thus by Proposition 2.1, we have the following result:

Corollary 2.2. *Let $R_i (1 \leq i \leq n)$ be rings, $T = \prod_{i=1}^n R_i$ and $M = M_1 \oplus \dots \oplus M_n$ be a T -module where each M_i is an R_i -module. Then ${}_R M$ is (completely) virtually semisimple if and only if ${}_T M$ is (completely) virtually semisimple.*

A non-zero submodule N of M is called *essential submodule* if N has non-zero intersection with every non-zero submodule of M and denoted by $N \leq_e M$. Each left R -module M has a singular submodule consisting of elements whose annihilators are essential left ideals in R . In set notation it is usually denoted as $Z(M)$ and M is called a *singular* (resp., *non-singular*) *module* if $Z(M) = M$ (resp., $Z(M) = 0$). Also, direct sum of simple submodules of ${}_R M$ is denoted by $\text{Soc}({}_R M)$. Direct sum of pairwise isomorphic submodules is called *homogenous components*. The following result shows that the study of virtually semisimple modules M with Dedekind finite $Z(M)$ reduces to the study of such modules when they are either singular or non-singular.

Proposition 2.3. *Let R be a ring and M be an R -module. Then:*

- (i) *If M is virtually semisimple such that $Z(M)$ is Dedekind finite. Then $M \cong W \oplus L$ where W is a singular virtually semisimple R -module and L is a non-singular virtually semisimple R -module.*
- (ii) *If every homogenous components of $\text{Soc}(M)$ is finitely generated. Then M is virtually semisimple if and only if $M \cong W \oplus L$ where W is a semisimple R -module and L is a virtually semisimple R -module with $\text{Soc}(L) = 0$.*

Proof. (i) This is obtained by parts (i) and (v) of Proposition 2.1.

(ii) By hypothesis, $\text{Soc}(M)$ is Dedekind finite and hence, $M = \text{Soc}(M) \oplus L$ when ${}_R M$ is virtually semisimple. Conversely, assume that $M = W \oplus L$ where ${}_R W$ is semisimple and ${}_R L$ is virtually semisimple with $\text{Soc}(L) = 0$. Note that $W = \text{Soc}(M)$ and let $N \leq {}_R M$.

Then there exists a submodule W' of M such that $\text{Soc}(N) \oplus W' = W$. Since $\text{Soc}(N) \leq N \leq W \oplus L$, so $N = \text{Soc}(N) \oplus T$ where T is a submodule of M . Since $\text{Soc}(M) = W$, so $\text{Soc}(T) = 0$ and hence T is embedded in L . Now, since L is virtually semisimple, so T is isomorphic to a direct summand of L . It follows that N is isomorphic to a direct summand of M . Thus M is a virtually semisimple R -module. \square

The following example shows that the hypothesis of “ $Z(M)$ is Dedekind finite” in Part (i), and “every homogenous components of $\text{Soc}(M)$ is finitely generated” in Part (ii) of Proposition 2.3 can not be relaxed.

Example 2.4. For $R = \mathbb{Z}_4$ and $M = \mathbb{Z}_4 \oplus (\bigoplus_{i=1}^{\infty} \mathbb{Z}_2)$, we have $Z({}_R M) = \text{Soc}({}_R M)$ is not a direct summand of ${}_R M$. Since R is a commutative Artinian principle ideal ring, every R -modules is a direct sums of cyclic modules (see [3, Result 1.3]). We note that since M is countable, all submodules of M are also countable. It follows that every submodule of M is isomorphic to $\mathbb{Z}_4 \oplus (\bigoplus_{j \in J} \mathbb{Z}_2)$ or $\bigoplus_{j \in J} \mathbb{Z}_2$ where J is an index set with $|J| \leq \aleph$, i.e., M is a virtually semisimple R -module.

A module is called a *uniform module* if the intersection of any two non-zero submodules is non-zero. The *Goldie dimension* of a module M , denoted by $u.\dim(M)$, is defined to be n if there exists a finite set of uniform submodules U_i of M such that $\bigoplus_{i=1}^n U_i$ is an essential submodule of M . If no such finite set of submodules exists, then $u.\dim(M)$ is defined to be infinity (in the literature, the Goldie dimension of M is also called the rank, the Goldie rank, the uniform dimension of M).

Next we need the following two lemmas.

Lemma 2.5. (See [9, Lemmas 15.3 and 15.8]) *If ${}_R M$ is a Noetherian module, then $K.\dim(M)$ is defined. If ${}_R M$ is a non-zero module with Krull dimension, then ${}_R M$ has a critical submodule.*

Lemma 2.6. (See [14, Lemma 6.2.5]) *If ${}_R M$ is finitely generated, then $K.\dim(M) \leq K.\dim({}_R R)$.*

In the following proposition, we investigate some finiteness conditions of virtually semisimple modules.

Proposition 2.7. *For a non-zero virtually semisimple module M , the following conditions are equivalent.*

- (1) M is finitely generated.
- (2) M is Noetherian.

(3) $\text{u.dim}(M) < +\infty$.

(4) $M \cong R/P_1 \oplus \dots \oplus R/P_n$ where $n \in \mathbb{N}$ and each P_i is a quasi prime left ideal of R such that R/P_i is a critical Noetherian R -modules.

In any of the above cases, $M \cong N$ for all $N \leq_e M$.

Proof. (4) \Rightarrow (1) and (1) \Rightarrow (2) are by the facts that direct summands and finite direct sums of finitely generated module are again finitely generated.

(2) \Rightarrow (3) is well-known for any module (see for instance [9, Corollary 5.18]).

(3) \Rightarrow (4). Assume that $\text{u.dim}(M)$ is finite and we set $\text{u.dim}(M) = n$ where $n \in \mathbb{N}$. Then there exist uniform cyclic independent submodules U_1, \dots, U_n of M such that $U_1 \oplus \dots \oplus U_n := N \leq_e {}_R M$. Since M is virtually semisimple, so $N \cong K$ where $K \oplus K' = M$ for some $K' \leq M$. Since $\text{u.dim}(K) = \text{u.dim}(M)$, so $\text{u.dim}(K') = 0$ (see [9, Corollary 5.21]). Thus $M \cong N$, which implies that ${}_R M$ is finitely generated, hence ${}_R M$ is Noetherian, as we see in the proof of (1) \Rightarrow (2). Now by Lemma 2.5, every non-zero submodule of M contains a non-zero cyclic critical R -submodule. Thus the condition (3) and our assumption imply that M is isomorphic to $V_1 \oplus \dots \oplus V_n$ where each V_i is a critical Noetherian R -module. Now assume that $V \cong R/P$ is a α -critical left R -module and $AB \subseteq P \subseteq A \cap B$ for some ideals A, B of R . If $A \not\subseteq P$ and $B \not\subseteq P$, then $\text{K.dim}(R/A) < \alpha$ and $\text{K.dim}(R/B) < \alpha$. On the other hand, B/P is a finitely generated left (R/A) -module and hence by 2.6, $\text{K.dim}(B/P) \leq \text{K.dim}(R/A) < \alpha$. This contradicts $\text{K.dim}(R/P) = \max\{\text{K.dim}(R/B), \text{K.dim}(B/P)\}$. Therefor P is quasi prime. The proof is complete. \square

It is easily to see that if $N \cong M$ for all $N \leq_e {}_R M$, then ${}_R M$ is virtually semisimple. The Proposition 2.7 shows that a finitely generated module ${}_R M$ is virtually semisimple if and only if $N \cong M$ for all $N \leq_e {}_R M$. Thus in this case, ${}_R M$ is *essentially compressible* in the sense of [17] (i.e., $M \hookrightarrow N$ for all $N \leq_e {}_R M$). Essentially compressible modules are *weakly compressible* in the sense of [22] (i.e., $N\text{Hom}_R(M, N) \neq 0$ for any submodule $0 \neq N \leq {}_R M$). We state below some results related to these concepts and then apply them to investigating the endomorphism ring of a virtually semisimple module. First we need the following proposition from several articles. We recall that a ring R is left hereditary if and only if every left ideal is projective.

Proposition 2.8. *The following statements hold.*

- (i) *Any essential compressible module is weakly compressible.*
- (ii) *If ${}_R M$ is a non-zero quasi projective 1-epi-retractable, then $\text{End}_R(M)$ is a principal left ideal ring if and only if ${}_R M$ is epi-retractable. In particular, ${}_R R^{(n)}$ is epi-retractable if and only if $M_n(R)$ is a principal left ideal ring.*
- (iii) *If ${}_R M$ is a quasi projective retractable, then $\text{End}_R(M)$ is a semiprime ring if and only*

if ${}_R M$ is weakly compressible.

(iv) Every semiprime principal left ideal ring is a left hereditary ring.

Proof. The part (i) is by Theorems 3.1, 2.2 of [17]. (ii) is obtained by [6, Theorem 2.2]. The part (iii) is [10, Theorem 2.6] and (iv) is Lemma 4 of [4]. \square

Let R be a ring and M be a left R -module. If X is an element or a subset of M , we define the *annihilator* of X in R by $\text{Ann}_R(X) = \{r \in R \mid rX = (0)\}$. In the case R is non-commutative and X is an element or a subset of an R , we define the *left annihilator* of X in R by $\text{l. Ann}_R(X) = \{r \in R \mid rX = (0)\}$ and the *right annihilator* of X in R by $\text{r. Ann}_R(X) = \{r \in R \mid Xr = (0)\}$.

Theorem 2.9. *Let M be a quasi projective finitely generated 1-epi-retractable R -module. Then M is virtually semisimple if and only if $\text{End}({}_R M)$ is a semiprime principal left ideal ring.*

Proof. Set $S := \text{End}({}_R M)$ and then we apply Proposition 2.8.

(\Rightarrow). By Proposition 2.8(ii), S is a principal left ideal ring. To show that S is also semiprime, we note that since ${}_R M$ is virtually semisimple, it is essentially compressible. Thus by Proposition 2.8(i), ${}_R M$ is weakly compressible and so by Proposition 2.8(iii), S is a semiprime ring.

(\Leftarrow). Assume that S is a semiprime principal left ideal ring. To show that ${}_R M$ is virtually semisimple, let $N \leq {}_R M$. Again by Proposition 2.8(ii), ${}_R M$ is epi-retractable and hence there exists a surjective R -homomorphism $f : M \rightarrow N$. By first isomorphism theorem, it is enough to show that $\text{Ker}(f) \leq^\oplus {}_R M$. Note that $\text{Hom}_R(M, \text{Ker}(f)) = \text{l. Ann}_S(f)$. Assume that $\varphi : S \rightarrow Sf$ with $\varphi(g) = gf$. Since now ${}_S(Sf)$ is projective by Proposition 2.8(iv), so the $\text{l. Ann}_S(f) = \text{Ker}(\varphi)$ is a direct summand of S and hence $\text{l. Ann}_S(f) = Se$ for some $e^2 = e \in S$. Clearly, $\text{Im}(e) \subseteq \text{Ker}(f)$ and so by the epi-retractable condition, there exists a surjective homomorphism $h : M \rightarrow \text{Ker}(f)$. So $\text{Ker}(f) = \text{Im}(h)$ for some $h \in S$. Thus $h \in \text{Hom}_R(M, \text{Ker}(f)) = Se$, which implies that $h(1 - e) = 0$. It follows that $h = he$ and hence $\text{Ker}(f) = \text{Im}(he) \subseteq \text{Im}(e)$. This shows that $\text{Im}(e) = \text{Ker}(f)$. It is easily see that $\text{Im}(e)$ is a direct summand of M and the proof is complete. \square

3 Structure of left virtually semisimple rings

In this section, we investigate the structure of (completely) left virtually semisimple rings and give a generalization of Wedderburn-Artin theorem. Meanwhile, we give an example to show that the left/right distinction cannot be removed and also we provide an example

of left virtually semisimple ring which is not completely. We shall first note that every (right) left virtually semisimple is principal (right) left ideal ring. Moreover, we have the following result duo to A. Goldie.

Lemma 3.1. (See [8, Theorem A and B]) *A semiprime principal left ideal ring is finite direct product of matrix rings over left Noetherian domains.*

The following lemma is also needed.

Lemma 3.2. [6, Proposition 2.5] *Let R be a left hereditary ring. Then every left free R -module is epi-retractable if and only if R is a principal left ideal ring.*

We investigate below the class of left (completely) virtually semisimple rings. We should point out that every set can be well-ordered (see for instance [11]).

Proposition 3.3. *The following statements hold.*

- (i) *A ring R is left (completely) virtually semisimple if and only if every (projective) free left R -module is (completely) left virtually semisimple.*
- (ii) *Let R be a ring Morita equivalent to a ring S . Then R is a left completely virtually semisimple if and only if S is so.*
- (iii) *The class of left virtually semisimple (resp., left completely virtually semisimple) rings is closed under finite direct products.*
- (iv) *Let R be a left completely virtually semisimple ring. Then for any semisimple R -module N and projective R -module P with $\text{Soc}({}_R P) = 0$, $N \oplus P$ is a completely virtually semisimple R -module.*

Proof. (i) One direction is clear. In view of Lemma 3.2 and Proposition 2.1(iii), we shall prove the left completely virtually semisimple case. Let $F = \bigoplus_{\alpha \in \Omega} Re_\alpha$ be a free R -module with basis $\{e_\alpha\}_{\alpha \in \Omega}$ and $P \leq {}_R F$. As the proof of Kaplansky's theorem [12, Theorem 2.24], we fix a well-ordering “ $<$ ” on the indexing set Ω . For any $\alpha \in \Omega$, let F_α (resp., G_α) be the span of the e_β 's with $\beta \leq \alpha$ (resp., $\beta < \alpha$). Then each $a \in P \cap F_\alpha$ has a unique decomposition $a = b + re_\alpha$ with $b \in G_\alpha$ and $r \in R$. The mapping $\varphi_\alpha : a \mapsto r$ maps $P \cap F_\alpha$ onto a left ideal U_α with kernel $P \cap G_\alpha$. By Proposition 2.1(iii), R is left hereditary and so ${}_R U_\alpha$ is projective. Thus φ_α splits, so we have

$$P \cap F_\alpha = (P \cap G_\alpha) \oplus A_\alpha$$

for some submodule A_α of $P \cap F_\alpha$ isomorphic to U_α . It can be checked that $P = \bigoplus_{\alpha \in \Omega} A_\alpha$. Hence $P \cong \bigoplus_{\alpha \in \Omega} U_\alpha$ where $U_\alpha = \varphi_\alpha(P \cap F_\alpha)$. It follows that if $Q \leq {}_R P$, then $Q \cong \bigoplus_{\alpha \in \Omega} V_\alpha$ where $V_\alpha \subseteq U_\alpha$. Since R is left completely virtually semisimple, so ${}_R U_\alpha$ is

virtually semisimple for any $\alpha \in \Omega$. Hence V_α is isomorphic to a direct summand of U_α ($\alpha \in \Omega$). This shows that ${}_R P$ is virtually semisimple. The proof is complete.

(ii) It follows by (i) (since virtually semisimplity and projectivity conditions are Morita invariants).

(iii) is by Corollary 2.2.

(iv) Assume that $K \leq N \oplus P$. We shall show that K is a virtually semisimple R -module. Since $\text{Soc}({}_R P) = 0$, we have $\text{Soc}({}_R(N \oplus P)) = N$ and hence $\text{Soc}({}_R K)$ is a direct summand of K . Thus there is a submodule K' such that $K = \text{Soc}({}_R K) \oplus K'$. Clearly, $K' \cap N = 0$ and so K' can be embedded in ${}_R P$. Now assume that $W \leq K$. By a similar argument, we have $W = \text{Soc}({}_R W) \oplus W'$ where W' embeds in K' . By part (i), K' is virtually semisimple R -module. Therefore, W' is isomorphic to a direct summand of K' , proving that K is a virtually semisimple R -module. \square

Several characterizations of left virtually semisimple rings are given below.

Theorem 3.4. *The following statements are equivalent for a ring R .*

- (1) R is a left virtually semisimple ring.
- (2) R is a semiprime principal left ideal ring.
- (3) R is a left hereditary and principal left ideal ring.
- (4) $R \cong \prod_{i=1}^k M_{n_i}(D_i)$ where each D_i is a (Noetherian) domain and every $M_{n_i}(D_i)$ is a principal left ideal ring.

Proof. (1) \Rightarrow (2). Assume that R is a left virtually semisimple ring. Then it is clear that R is a principal left ideal ring. We will to show R is semiprime. Assume that $I^2 = 0$ where I is an ideal of R and $L := \text{r. Ann}_R(I)$. Then $I \subseteq L$ and for each $0 \neq s \in R$, since $I(Is) = I^2 s = 0$, we conclude that either $s \in L$ or $Is \subseteq L$. This shows that L is an essential left ideal of R . Thus ${}_R R \stackrel{\varphi}{\cong} L$ by Proposition 2.7. We have $\varphi(I) = \varphi(IR) = I\varphi(R) = IL = 0$. Thus $I = 0$ because φ is a monomorphism. Therefore, R is a semiprime principal left ideal ring.

(2) \Rightarrow (3) is by Proposition 2.8(iv).

(3) \Rightarrow (1). Since R is a principal left ideal ring, so ${}_R R$ is epi-retractable. Thus by Proposition 2.1(iii), R is a left virtually semisimple ring.

(2) \Rightarrow (4) is by Lemma 3.1.

(4) \Rightarrow (2) is by the fact that the class of semiprime principal left ideal rings is closed under finite direct products. \square

Remark 3.5. *The integers k and n_1, \dots, n_k in the Theorem 3.4 are uniquely determined by R because of the following lemma.*

Lemma 3.6. *Let $\{D_i\}_{i=1}^k$ and $\{D'_j\}_{j=1}^r$ be two families of left Noetherian domains such that $\prod_{i=1}^k M_{n_i}(D_i) \cong \prod_{j=1}^r M_{k_j}(D'_j)$ as ring. Then $r = k$ and there exists a permutation ξ on set $\{1, \dots, r\}$ such that for each $i \in \{1, \dots, r\}$, $n_i = k_{\xi(i)}$.*

Proof. Let $R = \prod_i M_{k_i}(D_i)$. Thus R is a semiprime left Noetherian. Let $Q = Q(R)$ be the classical left quotient ring of R . By our assumption, we can conclude $Q = \prod_i M_{n_i}(Q_i) \cong \prod_j M_{k_j}(Q'_j)$ where Q_i and Q'_j are division rings. Hence the result is obtained by Wedderburn-Artin Theorem [13, Theorem 3.5]. \square

In view of the above characterization 3.4(iv) of a left virtually semisimple ring, the following natural question arises: “Are the domains in Theorem 3.4(iv) unique up to isomorphism?”. The following example shows that the answer is negative in general.

Example 3.7. Let $D = A_1(F)$, the first Weyl algebra over a field F with characteristic zero. It is known that D is a simple Noetherian domain (see for instance [14, Chapter 1, §3, Theorem 1.3.5]). If I is a non-zero left ideal of $A_1(F)$ then by [19, Theorem 3(i)], $I \oplus I \cong D \oplus D$ and so $M_2(D) \cong M_2(\text{End}_D(I))$. While by [16, Proposition 1], D is not isomorphic to the ring $\text{End}_D(I)$.

In the sequel, we show that the answer of the above question is positive when R is a left completely virtually semisimple ring. In this case, R is determined by a set of matrix rings over principal left ideal domains and the size of matrix rings. Firstly, we give an example to show that there exists a left virtually semisimple ring which is not a left completely virtually semisimple ring.

Example 3.8. Let $R = A_1(F)$, the first Weyl algebra over a field F with characteristic zero, and let $S = M_2(R)$. By [14, Example 7.11.8], R is not a (left) principal ideal domain and hence S can not be a left completely virtually semisimple by Proposition 3.3(ii) while S is a semiprime principal ideal ring (virtually semisimple ring) by ([14, Chapter 7, §11, Corollary 7.11.7]).

The following example shows that for a ring the (completely) virtually semisimple property is not symmetric.

Example 3.9. In [5, Section 4], it is given an example of principal right ideal domain R which is not left hereditary. Thus by Theorem 3.4, we deduce that R is right (completely) virtually semisimple which is not left virtually semisimple.

The following provide an example of a simple ring R such that R is a left and a right (completely) virtually semisimple ring, but it is not semisimple.

Example 3.10. Let A be a field and $\varphi : A \rightarrow A$ a ring automorphism such that $\varphi^n \neq 1$ for every natural number n . If $R = A[x, x^{-1}, \varphi]$, then by [18, Proposition 4.7], R is a simple principal left and right ideal domain that is not a division ring. But it is clear that R is a left and right (completely) virtually semisimple ring.

A ring R is said to be M_n -unique if, for any ring S , $M_n(R) \cong M_n(S)$ implies that $R \cong S$ (see for instance [12, §17.C]). Let \mathcal{C} be a class of R -modules. Following [12, §17C], we say that \mathcal{C} satisfies *weak n -cancellation* if, for any P, Q in \mathcal{C} , the condition $P^{(n)} \cong Q^{(n)}$ implies that $\text{End}_R(P) \cong \text{End}_R(Q)$. An R -module P is said to be a *generator* for $R\text{-Mod}$ if R is a direct summand $\bigoplus_{\lambda \in \Lambda} P$ for some finite index set Λ .

We need the following lemmas.

Lemma 3.11. (See [12, Theorem 17.29]) *For any ring S , and any given integer $n \geq 1$, the following two statements are equivalent.*

- (1) *Any ring T Morita-equivalent to S is M_n -unique.*
- (2) *The class of finitely generated projective generators in $S\text{-Mod}$ satisfies the weak n -cancellation property.*

Lemma 3.12. (Kaplansky's Theorem [12, Theorem 2.24]). *Let R be a left hereditary ring. Then every submodule P of a free left R -module F is isomorphic to a direct sum of left ideals of R .*

We are now in a position to prove the following generalization of the Wedderburn-Artin theorem.

Theorem 3.13. *Let R be a ring. Then R is a left completely virtually semisimple ring if and only if $R \cong \prod_{i=1}^k M_{n_i}(D_i)$ where $k, n_1, \dots, n_k \in \mathbb{N}$ and each D_i is a principal left ideal domain. Moreover, the integers k, n_1, \dots, n_k and the principal left ideal domains D_1, \dots, D_k are uniquely determined (up to isomorphism) by R .*

Proof. (\Rightarrow). Assume that R is a left completely virtually semisimple ring. Then by Theorem 3.4(iv), $R \cong \prod_{i=1}^k M_{n_i}(D_i)$ where each D_i is a domain and every $M_{n_i}(D_i)$ is a principal left ideal ring. Since R is a left completely virtually semisimple ring, so $M_{n_i}(D_i)$ is a left completely virtually semisimple ring for each i ($1 \leq i \leq k$). By Proposition 3.3(ii), each D_i is left completely virtually semisimple, i.e., each D_i is a principal left ideal domain. Hence in view of Lemma 3.11 and Theorem 3.4, in order to prove that each D_i is uniquely determined, it is enough to check that any principal left ideal domain D satisfies the condition(ii) of Lemma 3.11.

Suppose that P and Q are finitely generated generators in $D\text{-Mod}$ and $P^{(n)} \cong Q^{(n)}$ for $n \geq 1$. By Lemma 3.12, every projective D -module is free. Thus $P \cong D^{(r)}$ and

$Q \cong D^{(s)}$ for suitable integer numbers $r, s \geq 1$. Therefore, $D^{(nr)} \cong D^{(ns)}$ and since D is left Noetherian, so by the invariant basis number property on D (see for instance [12, Page 17]), we conclude that $r = s$ and hence ${}_D P \cong {}_D Q$.

(\Leftarrow). Clearly every principal left ideal domains is completely virtually semisimple. Thus the implication is obtained by Proposition 2.1. \square

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